The Unified Transform Method: An unexpected extension of classical works of d'Alembert, Fourier and Laplace

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UNIVERSITY OF CAMBRIDGE | GIANNA ANGELOPOULOS PROGRAMME FOR SCIENCE, TECHNOLOGY AND INNOVATION

1747, d' Alembert and Euler: Separation of Variables

1807, Fourier: Transforms

1814, Cauchy: Analyticity

1828, Green: Green's Representations

1845, Kelvin: Images

Fourier (spectral) Space
Transforms

Physical Space
Green's Integral Representations
Method of Images

New Method
Integral Representations in the Fourier Spece
Invariance of Global Relation and Jordan's Lemma
http://unifiedmethod.azurewebsites.net/
https://en.wikipedia.org/wiki/Fokas_method

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- B. Deconinck, T. Trogdon and V. Vasan, The Method of Fokas for Solving Linear Partial Differential Equations, SIAM Review 56, 159-186 (2014)
- A.S. Fokas and E. Kaxiras, Modern Mathematical Methods for Scientists and Engineers, in press

Linear Evolution PDEs on the Half-Line
Classical methods

The heat equation

$$u_t = u_{xx}, \quad 0 < x < \infty, \;\; 0 < t < T, \quad T > 0,$$

Initial condition:

$$u(x,0) = u_0(x), \;\; 0 < x < \infty,$$

Boundary condition:

$$u(0,t) = g_0(t), \;\; 0 < t < T.$$

Classical Sine Transform method:

$$\mathrm{PDE}[u(x,t)] \xrightarrow[Direct]{ST} \mathrm{ODE}[\hat{u}(\lambda,t)] \xrightarrow{Solve} \hat{u}(\lambda,t) \xrightarrow[Inverse]{ST} u(x,t)$$

The classical solution:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty e^{-\lambda^2 t}\sin(\lambda x)\Big[\int_0^\infty \sin(\lambda\xi)u_0(\xi)d\xi - \lambda\int_0^t e^{\lambda^2 s}g_0(s)ds\Big]d\lambda.$$

Disadvantages of the Traditional Transforms

1. Lack of uniform convergence at the boundaries (for inhomogeneous boundary conditions).
2. Not straightforward to verify that the solution representation actually solves the given BVP.
3. Not suitable for numerical computations.
4. Requires separability of PDE-domain-BCs. For example, cannot be applied to
$$\int_0^\infty K(x,t)u(x,t)dx = g(t).$$
5. Difficult to obtain appropriate transform.
6. Traditional transforms exist only for a very limited class of problems.

There are no x-transforms for the diffusion-convection equation
$$u_t = u_{xx} + u_x,$$
or the Stokes (written by Sir George Stokes)
$$u_t + u_x + u_{xxx} = 0.$$

Summary of Fokas Method

"Green + Fourier + Cauchy"

Three ingredients:

1. Global Relation - equation coupling transforms of boundary values.
2. Integral representation - solution given as integral of *transforms* of boundary values. ("Green + Fourier").
3. Symmetries which leave transforms of the boundary values invariant. Use information given by 1 to *either* find unknown boundary values *or* their contributions to 2.

In all but the simplest cases need to consider transforms of boundary values as functions in $\mathbb{C}$ ("Cauchy").

The heat equation via the Fokas Method

The heat equation can be written as a family of divergence forms:

$$\left(e^{-i\lambda x+\lambda^2 t}u\right)_t - \left(e^{-i\lambda x+\lambda^2 t}(u_x + i\lambda u)\right)_x = 0, \quad \lambda \in \mathbb{C}.$$

Green's Theorem: $$\int_{\partial\Omega} e^{-i\lambda x+\lambda^2 t}[u dx + (u_x + i\lambda u)dt] = 0.$$

For the half line the Global Relation is

$$e^{\lambda^2 t}\hat{u}(\lambda, t) = \hat{u}_0(\lambda) - \tilde{g}_1(\lambda^2, t) - i\lambda\tilde{g}_0(\lambda^2, t), \quad \mathrm{Im}\lambda \le 0,$$

where

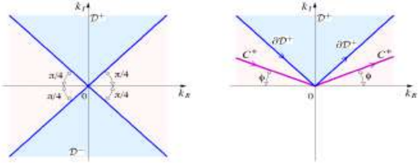
$$\hat{u}(\lambda, t) = \int_0^\infty e^{-i\lambda x}u(x,t)dx, \qquad \hat{u}_0(\lambda) = \int_0^\infty e^{-i\lambda x}u_0(x)dx, \ \ \mathrm{Im}\lambda \le 0,$$

$$\tilde{g}_j(\lambda, t) = \int_0^t e^{\lambda\tau}g_j(\tau)d\tau, \quad \text{with } g_1(t) = u_x(0,t), \ g_0(t) = u(0,t), \ t > 0.$$

First use of the GR:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(\lambda)d\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\left[\tilde{g}_1(\lambda^2,t) - i\lambda\tilde{g}_0(\lambda^2,t)\right]d\lambda.$$

The second integral can be deformed to the curve ∂D^+, defined by $\mathrm{Re}(\lambda^2)=0,\ \mathrm{Im}\lambda>0$.



Second use of the GR:

$$e^{\lambda^2 t}\hat{u}(-\lambda,t) = \hat{u}_0(-\lambda) - \tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t), \quad \mathrm{Im}\lambda \geq 0.$$

The solution takes the form

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[\hat{u}_0(-\lambda) + 2i\lambda\tilde{g}_0(\lambda^2,t)\right]d\lambda.$$

Also, we can replace $\tilde{g}_0$ with $G_0(\lambda^2) = \int_0^T e^{\lambda^2 s}g_0(s)ds, \quad \lambda\in\mathbb{C}.$

Consistent with the Ehrenpreis Principle:

$$u(x,t) = \int_L e^{i\lambda x-\lambda^2 t}d\rho(\lambda).$$

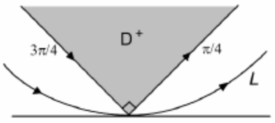
Numerical Implementation

Consider the heat equation with

$$u_0(x) = e^{-ax}, \quad g_0(t) = \cos(bt), \qquad a > 0, \; b > 0.$$

Then,

$$u(x,t) = \int_L e^{i\lambda x - \lambda^2 t} \left[\frac{1}{i\lambda + a} + \frac{1}{i\lambda - a} + i\lambda \left(\frac{1}{\lambda^2 + ib} + \frac{1}{\lambda^2 - ib} \right) \right] \frac{d\lambda}{2\pi}$$
$$- \int_L e^{i\lambda x} i\lambda \left(\frac{e^{ibt}}{\lambda^2 + ib} + \frac{e^{-ibt}}{\lambda^2 - ib} \right) \frac{d\lambda}{2\pi}.$$



de Barros, Colbrook and Fokas. A hybrid analytical-numerical method for solving advection-dispersion problems on a half-line. International Journal of Heat and Mass Transfer. (2019)

The last term can be evaluated explicitly, via Residue theory:

$$u(x,t) = \int_L e^{i\lambda x - \lambda^2 t} V(\lambda; a, b) \frac{d\lambda}{2\pi} + e^{-x\sqrt{\frac{b}{2}}} \cos\left(bt - x\sqrt{\frac{b}{2}}\right),$$

with $V(\lambda; a, b) = \frac{1}{i\lambda + a} + \frac{1}{i\lambda - a} + i\lambda\left(\frac{1}{\lambda^2 + ib} + \frac{1}{\lambda^2 - ib}\right)$.
Verification: Evaluating the above equation at $x = 0$ yields

$$u(0,t) = \int_L e^{-\lambda^2 t} V(\lambda; a, b) \frac{d\lambda}{2\pi} + \cos(bt) = \cos(bt).$$

By deforming the contour L to the real line and observing that $V(\lambda; a, b)$ is an odd function of λ, the above integral vanishes. Evaluating the previous equation at $t = 0$ yields

$$u(x,0) = \int_C e^{i\lambda x}\left(\frac{1}{i\lambda + a} + \frac{1}{i\lambda - a}\right)\frac{d\lambda}{2\pi}.$$

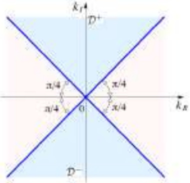
Then, Cauchy's theorem yields

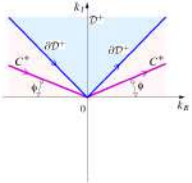
$$u(x,0) = \frac{2\pi i}{2\pi} e^{i(ia)x} \frac{1}{i} = e^{-ax}.$$

Solutions of linear second and third order PDES

$$u_t = u_{xx}, \qquad x > 0,\ t > 0$$

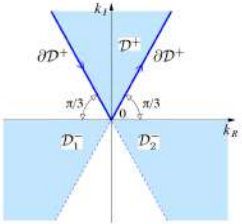
$$u(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{u}_0(\lambda)\frac{d\lambda}{2\pi} - \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left[\hat{u}_0(-\lambda) + 2i\lambda\tilde{g}_0(\lambda^2, t)\right]\frac{d\lambda}{2\pi}.$$





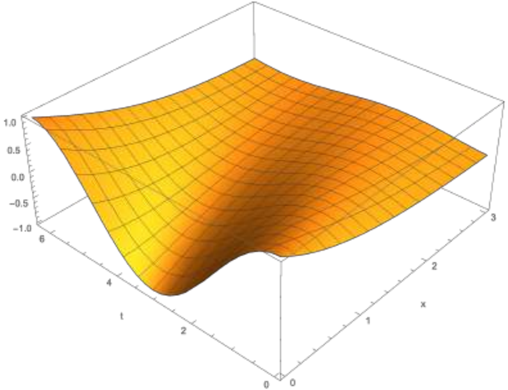
$$u_t + u_{xxx} = 0, \qquad x > 0,\ t > 0$$

$$u(x,t) = \int_{-\infty}^{+\infty} e^{i\lambda x + i\lambda^3 t}\hat{u}_0(\lambda)\frac{d\lambda}{2\pi}$$
$$+ \int_{\partial D^+}\left\{e^{i\lambda x + i\lambda^3 t}\left[e^{\frac{2i\pi}{3}}\hat{u}_0\left(e^{\frac{2i\pi}{3}}\lambda\right) + e^{\frac{4i\pi}{3}}\hat{u}_0\left(e^{\frac{4i\pi}{3}}\lambda\right)\right.\right.$$
$$\left.\left. - 3\lambda^2\tilde{g}_0(-i\lambda^3, t)\right]\right\}\frac{d\lambda}{2\pi}.$$



Heat Equation - Half Line
Dirichlet Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0, +\infty), \;\; t \in (0, T), \\ u(x,0) = e^{-x}, & x \in (0, +\infty) \\ u(0,t) = \cos t, & t \in (0, T). \end{cases}$$



Fokas & Kaxiras, Modern Mathematical Methods for Scientists and Engineers, in press

Heat Equation - Half Line
Oblique Robin Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0, +\infty), \ \ t \in (0, T), \\ u(x, 0) = xe^{-4x}, & x \in (0, +\infty) \\ u_t(0, t) - 2u_x(0, t) + u(0, t) = \sin(5t), & t \in (0, T). \end{cases}$$

Solution:

$$u(x, t) = \int_L V(\lambda, x, t) \frac{d\lambda}{2\pi} + v(x, t),$$

where

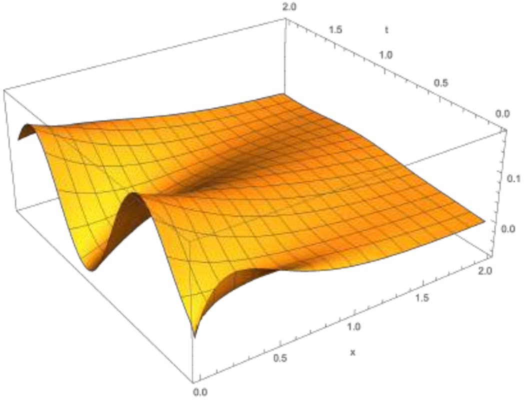
$$V(\lambda, x, t) = e^{i\lambda x - \lambda^2 t} \left(\frac{\frac{10i\lambda}{\lambda^4 + 25} + \frac{(\lambda - i)^2}{(\lambda + 4i)^2}}{(\lambda + i)^2} + \frac{1}{(4 + i\lambda)^2} \right),$$

and

$$v(x, t) = e^{-\sqrt{\frac{5}{2}}x} \frac{\left(\sqrt{10} + 1\right) \sin\left(5t - \sqrt{\frac{5}{2}}x\right) - \left(\sqrt{10} + 5\right) \cos\left(5t - \sqrt{\frac{5}{2}}x\right)}{\left(\sqrt{10} + 6\right)^2}.$$

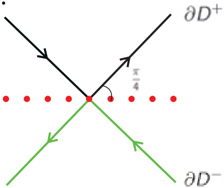
Heat Equation - Half Line
Oblique Robin Problem

Illustration of the solution:



Heat Equation - Finite Interval
General Dirichlet Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0, L),\ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in (0, L) \\ u(0, t) = g_0(t), \quad u(L, t) = h_0(t), & t \in (0, T). \end{cases}$$



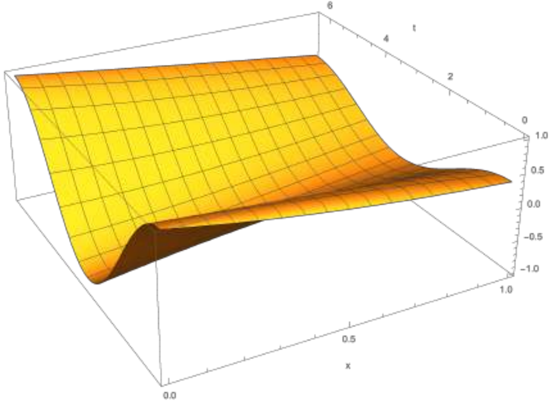
$$u(x, t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) \frac{d\lambda}{2\pi}$$

$$- \int_{\partial D^+} \frac{2e^{-\lambda^2 t}}{e^{i\lambda L} - e^{-i\lambda L}} \Bigg\{ \sin(\lambda x) \left[i e^{i\lambda L} \hat{u}_0(\lambda) - 2\lambda \tilde{h}_0\left(\lambda^2 t\right) \right]$$

$$+ \sin[\lambda(L - x)] \left[i \hat{u}_0(-\lambda) - 2\lambda \tilde{g}_0\left(\lambda^2 t\right) \right] \Bigg\} \frac{d\lambda}{2\pi}$$

Heat Equation - Finite Interval
Specific example on Dirichlet Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0,1),\ \ t \in (0, T), \\ u(x,0) = e^{-x}, & x \in (0,1) \\ u(0,t) = \cos t, \quad u(1,t) = \frac{1}{e}\cos t, & t \in (0, T). \end{cases}$$



The wave equation

$$u_{tt} - u_{xx} = 0, \qquad x > 0, \quad t > 0.$$

Dirichlet conditions

$$\begin{aligned} u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad & x > 0, \\ u(0,t) = g_0(t), \qquad & t > 0. \end{aligned}$$

Solution in the Fourier plane:

$$\begin{aligned} u(x,t) = & \int_{-\infty}^{\infty} e^{ikx} \left[\frac{\sin(kt)}{k} \hat{u}_1(k) + \cos(kt)\hat{u}_0(k) \right] \frac{dk}{2\pi} \\ & - \int_{-\infty}^{\infty} e^{ikx} \left[\frac{\sin(kt)}{k} \hat{u}_1(-k) + \cos(kt)\hat{u}_0(-k) + 2ik\check{g}_0(k,t) \right] \frac{dk}{2\pi}, \end{aligned}$$

Solution in the physical plane:

$$u(x,t) = \frac{1}{2} u_0(x+t) + \frac{1}{2} \int_{|x-t|}^{x+t} u_1(\xi) d\xi + \begin{cases} \frac{1}{2} u_0(x-t), & x > t, \\ g_0(t-x) - \frac{1}{2} u_0(t-x), & x < t. \end{cases}$$

Laplace equation on a polygon

Let $u(x, y)$ satisfy the Laplace equation in the interior of a convex polygon characterized by $z_1, ..., z_n$. Then $u_z(z)$ satisfies

$$u_z(z) = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{l_j} e^{i\lambda z} \hat{u}_j(\lambda) d\lambda, \qquad u_z = \frac{1}{2}(u_x - iu_y), \qquad z = x + iy,$$

where $\hat{u}_j(\lambda)$ is defined by

$$\hat{u}_j(\lambda) = \int_{z_j}^{z_{j+1}} e^{-i\lambda z} \left[\frac{\partial u}{\partial N} + \lambda u \frac{dz}{ds} \right] ds, \quad j = 1, ..., n, \qquad \lambda \in \mathbb{C}.$$

and the rays $\{l_j\}_1^n$ are defined by

$$l_j = \rho e^{i\theta_j}, \quad 0 < \rho < \infty, \quad \theta_j = -\arg(z_{j+1} - z_j), \quad j = 1, ..., n.$$

Furthemore, the following GR is valid:

$$\sum_{j=1}^{n} \hat{u}_j(\lambda) = 0, \qquad \lambda \in \mathbb{C}.$$

Numerical approach

The numerical solution of the global relation for determining the unknown boundary values involves the following two steps.

1. *Expand* the functions $\{u_j\}_1^n$ and $\left\{\frac{\partial u_j}{\partial w}\right\}_1^u$ in terms of N basis functions denoted by $\{S_l(t)\}_0^{N-1}$:

$$u_j \approx \sum_{l=0}^{N-1} a_l^j S_l(t), \quad \frac{\partial u_j(t)}{\partial \omega} \approx \sum_{l=0}^{N-1} b_l^j S_l(t), \quad j = 1, 2, \ldots, n.$$

Choosing S_l to be the Legendre polynomials, denoted by P_l, the relevant Fourier transform can be computed explicitly,

$$\hat{P}_l(\lambda) = \int_{-1}^{1} e^{-i\lambda t} P_l(t) dt = i \sum_{k=0}^{l} \frac{(l+k)!}{(l-k)!k!} \left[\frac{(-1)^{l+k} e^{i\lambda} - e^{-i\lambda}}{(2i\lambda)^{k+1}}\right].$$

2. The global relation and its complex conjugate yield two equations involving the constants a_l^j and b_l^j. By evaluating these equations at appropriately chosen values of λ called *collocation points*, we can solve for the unknown coefficients.

The case of the Square

Consider the Laplace equation in the interior of the square with corners

$$z_1 = -1 + i, \quad z_2 = -1 - i, \quad z_3 = 1 - i, \quad z_4 = 1 + i.$$

The approximate global relation yields

$$\hat{u}_1(\lambda) + \hat{u}_2(\lambda) + \hat{u}_3(\lambda) + \hat{u}_4(\lambda) = 0, \quad \lambda \in \mathbb{C},$$

where

$$\hat{u}_1(\lambda) \approx -e^{i\lambda} \sum_{l=0}^{N-1} \left[i\lambda a_1^l \hat{P}_l(\lambda) + b_1^l \hat{P}_l(\lambda) \right],$$

$$\hat{u}_2(\lambda) \approx e^{-\lambda} \sum_{l=0}^{N-1} \left[\lambda a_2^l \hat{P}_l(-i\lambda) + b_2^l \hat{P}_l(-i\lambda) \right],$$

$$\hat{u}_3(\lambda) \approx e^{-i\lambda} \sum_{l=0}^{N-1} \left[i\lambda a_3^l \hat{P}_l(\lambda) + b_3^l \hat{P}_l(\lambda) \right],$$

$$\hat{u}_4(\lambda) \approx e^{\lambda} \sum_{l=0}^{N-1} \left[-\lambda a_4^l \hat{P}_l(-i\lambda) + b_4^l \hat{P}_l(-i\lambda) \right].$$

For a given side, choose λ in such a way that for the given side we obtain the usual Fourier transform of the Legendre functions, whereas the contribution from the remaining sides vanishes as $\lambda \to \infty$ (it can be shown that for a *convex* polygon such a choice is always possible). In fact we choose to be on the "complement" of the rays l_j.

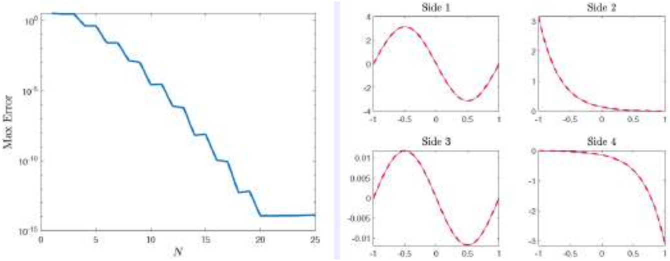
- side 1. Multiply the approximate global relation by $e^{-i\lambda}$ and then let $\lambda = -i\rho$, $\rho > 0$.
- side 2. Multiply by e^{λ} and then let $\lambda = -\rho$, $\rho > 0$.
- side 3. Multiply by $e^{i\lambda}$ and then let $\lambda = i\rho$, $\rho > 0$.
- side 4. Multiply by $e^{-\lambda}$ and then let $\lambda = \rho$, $\rho > 0$.

For ρ we can use the discrete values $\rho = \frac{R}{M}m$, $m = 1, 2, \ldots, M$, $R > 0$, where R/M determines how close are the collocation points.

Numerical experiments suggest the following rules for low condition number:

$$R = 2M, \qquad M = Nn.$$

Dirichlet-to-Neumann map: Knowing a_j^l and evaluating the *Legendre expansion* on the *collocation points* in the approximate global relation we can determine the coefficients b_j^l numerically. For real-valued u, we also evaluate the Schwartz conjugate of the global relation at the complex conjugates of the suggested collocation points. The result is an overdetermined linear system, which is inverted in the least squares sense.



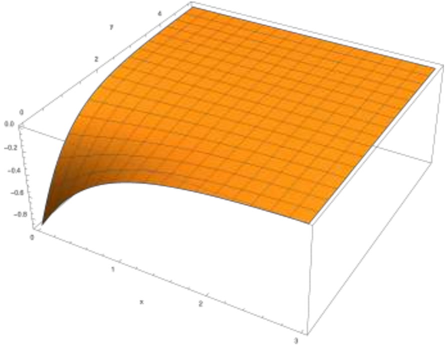
The plots above show the output of this procedure. The left panel shows the convergence of the method and the right panel typical computed boundary values. We see exponential convergence, since this simple example has no corner singularities.

Modified Helmholtz equation on the quarter plane

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) - u(x,y) = 0, & x > 0, \quad y > 0, \\ u_x(0,y) = e^{-2y}, \quad y > 0 \quad \text{and} \quad u_y(x,0) = e^{-3x}, & x > 0. \end{cases}$$

We obtain the following solution

$$u(x,y) = \int_0^{\infty e^{i\frac{\pi}{4}}} e^{\frac{i}{2}\left[\lambda(x+iy) - \frac{x-iy}{\lambda}\right]} \left(\frac{8\lambda}{\lambda^4 - 14\lambda^2 + 1} - \frac{12\lambda}{\lambda^4 + 34\lambda^2 + 1} \right) \frac{d\lambda}{\pi}.$$



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Water waves - The potential equation

Let the domain Ω_f be defined by

$$\Omega_f = \{-\infty < x < \infty, ; \quad -h < y < \eta(x,t); \quad t > 0\}.$$

Irrotational: vorticitity $\gamma = V_x - U_y = 0$.
Let ϕ denote the velocity potential, i.e. $\nabla\phi = (U, V)$.
The two unknown functions $\eta(x,t)$ and $\phi(x,y,t)$ satisfy the following equations:

$$\Delta\phi = 0 \;\; \text{in} \;\; \Omega_f,$$

$$\phi_y = 0 \;\; \text{on} \;\; y = -h,$$

$$\eta_t + \phi_x\eta_x = \phi_y \;\; \text{on} \;\; y = \eta,$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g\eta = 0 \;\; \text{on} \;\; y = \eta,$$

where g is the gravitational acceleration, and h is the constant unperturbed fluid depth.

The Global Relation

Introduce q which denotes the value of ϕ on the free surface, i.e.,

$$q(x,t) = \phi\left(x, \eta(x,t), t\right), \quad -\infty < x < \infty, \quad t > 0.$$

The global relation under appropriate transformations, yields a novel non-local equation coupling η and q,

$$\int_{-\infty}^{\infty} e^{ikx} \left\{ i\eta_t \cosh[k(\eta+h)] + q_x \sinh[k(\eta+h)] \right\} dx = 0, \;\; k \in \mathbb{R}, \;\; t > 0.$$

Furthermore, under the additional assumption of zero surface tension, equation the Bernoulli's law is rewritten

$$q_t + \frac{1}{2}\left(q_x\right)^2 + g\eta - \frac{\left[\eta_t + q_x\eta_x\right]^2}{2\left[1 + \left(\eta_x\right)^2\right]} = 0, \quad -\infty < x < \infty, \quad t > 0.$$

Periodic travelling waves

The above equation is a quadratic equation for q', thus we get

$$q' = -c + \sqrt{[1+(\eta')^2](c^2 - 2g\eta)}.$$

Let $\Omega_p = \{-L < x < L,; \quad -h < y < \eta(x,t); \quad t > 0\}$, and η and ϕ are $2L$-periodic functions in x.
We let $L = \pi$. Then we get

$$\int_{-\pi}^{\pi} e^{ikx} \left[\left(1 - \sqrt{(1+(\eta')^2)\left(1 - \frac{2g}{c^2}\eta\right)} \right) \sinh(k(\eta+h)) \right.$$
$$\left. + i\eta' \cosh[k(\eta+h)] \right] dx = 0, \qquad \text{for all } k \in \mathbb{Z}$$

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Additional Applications

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From Nonlinear to Linear

$$u(x,t): \qquad iu_t + u_{xx} = 0.$$

Rewrite it as the compatibility condition $\left(M_x\right)_t - \left(M_t\right)_x = 0$, namely

$$\left[e^{i\lambda x + i\lambda^2 t} u\right]_t - i\left[e^{i\lambda x + i\lambda^2 t}(u_x + i\lambda u)\right]_x = 0, \quad \lambda \in \mathbb{C}.$$

By defining $M = e^{i\lambda x + i\lambda^2 t}\mu$, the associated Lax pair consists of the following two linear equations:

$$\begin{aligned} &\mu_x + i\lambda\mu = u, \\ &\mu_t + i\lambda^2\mu = iu_x + \lambda u, \qquad \lambda \in \mathbb{C}. \end{aligned}$$

Compare with the classical separation of variables:

$$u(x,t) = X(x;\lambda)T(t;\lambda) \quad \Longrightarrow \quad \begin{cases} X'' + \lambda^2 X = 0, \\ T' - i\lambda^2 T = 0. \end{cases}$$

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Asymptotics of the Neumann value for t-periodic data

Example

$$q(0,t) = ae^{i\omega t} + o(1), \quad a, \omega \in \mathbb{R}, \quad t \to \infty$$

Claim: For $\lambda = -1$ (focusing NLS)

$$q_x(0,t) = ce^{i\omega t} + o(1), \quad t \to \infty,$$
$$c = a\sqrt{\omega - a^2}, \qquad \omega \geq a^2,$$

or

$$c = ia\sqrt{|\omega| + 2a^2} \qquad \omega \leq -6a^2.$$

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III. The NLS equation - The periodic problem

The NLS equation and its Lax Pair

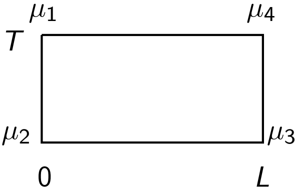
$$iq_t + q_{xx} - 2\lambda q|q|^2 = 0, \qquad \lambda = \pm 1. \tag{1}$$

The NLS equation (1) has a Lax pair given by

$$\mu_x + ik\hat{\sigma}_3\mu = Q\mu, \qquad \mu_t + 2ik^2\hat{\sigma}_3\mu = \tilde{Q}\mu, \tag{2}$$

$$Q = \begin{pmatrix} 0 & q \\ \lambda\bar{q} & 0 \end{pmatrix}, \qquad \tilde{Q} = 2kQ - iQ_x\sigma_3 - i\lambda|q|^2\sigma_3, \tag{3}$$

and $\hat{\sigma}_3\mu = [\sigma_3, \mu]$, with $\sigma_3 = \mathrm{diag}(-1, 1)$.



THE SPECTRAL FUNCTIONS

$$s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \lambda\overline{b(\bar{k})} & a(k) \end{pmatrix}, \quad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \lambda\overline{B(\bar{k})} & A(k) \end{pmatrix}, \quad S_L(k) = \begin{pmatrix} \overline{\mathcal{A}(\bar{k})} & \mathcal{B}(k) \\ \lambda\overline{\mathcal{B}(\bar{k})} & \mathcal{A}(k) \end{pmatrix},$$

where

$s(k) = \mu_3(0,0,k), \quad S(k) = \mu_1(0,0,k), \quad S_L(k) = \mu_4(L,0,k).$
The entries of the matrices are related by the global relation:

$$(a\mathcal{A} + \lambda\bar{b}e^{2ikL}\mathcal{B})B - (b\mathcal{A} + \bar{a}e^{2ikL}\mathcal{B})A = e^{4ik^2T}c^+(k), \qquad k \in \mathbb{C}, \quad (4)$$

where

$$c^+(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{2ikL}}{k}\right), \qquad k \to \infty, \ k \in \mathbb{C}. \qquad (5)$$

The above functions satisfy the unit determinant relations

$$a\bar{a} - \lambda b\bar{b} = 1, \qquad A\bar{A} - \lambda B\bar{B} = 1, \qquad \mathcal{A}\bar{\mathcal{A}} - \lambda\mathcal{B}\bar{\mathcal{B}} = 1. \quad (6)$$

The functions a and b satisfy

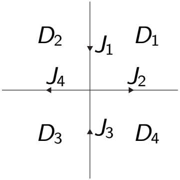
$$a(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{2ikL}}{k}\right), \quad b(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{2ikL}}{k}\right), \qquad k \to \infty,$$

whereas the functions A and B satisfy

$$A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \qquad k \to \infty.$$

THE RH PROBLEM FOR THE PERIODIC CASE

$$q(0,t) = q(L,t) \quad \text{and} \quad q_x(0,t) = q_x(L,t) \quad \text{for} \quad t \in [0,\, T]. \quad (7)$$



In this case $\mathcal{A} = A$ and $\mathcal{B} = B$. The jump matrices:

$$J_1 = \begin{pmatrix} \delta/d & -Be^{2ikL}e^{-2i\theta} \\ \lambda\bar{B}e^{2i\theta}/(d\alpha) & a/\alpha, \end{pmatrix} \qquad J_2 = \begin{pmatrix} 1 & -\beta e^{-2i\theta}/\bar{\alpha} \\ \lambda\bar{\beta}e^{2i\theta}/\alpha & 1/(\alpha\bar{\alpha}) \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \bar{\delta}/\bar{d} & -Be^{-2i\theta}/(\bar{d}\bar{\alpha}) \\ \lambda\bar{B}e^{-2ikL}e^{2i\theta} & \bar{a}/\bar{\alpha} \end{pmatrix}, \quad J_4 = J_3 J_2^{-1} J_1, \qquad (8)$$

where

$$\alpha = aA + \lambda\bar{b}Be^{2ikL}, \quad \beta = bA + \bar{a}Be^{2ikL}, \quad d = a\bar{A} - \lambda b\bar{B}, \quad \delta = \alpha\bar{A} - \lambda\beta\bar{B}.$$

The global relation (4) becomes

$$\lambda\bar{b}e^{2ikL}B^2 + (a - \bar{a}e^{2ikL})AB - bA^2 = e^{4ik^2T}c^+. \qquad (9)$$

THE NEW RH PROBLEM INVOLVING Γ

$$\Gamma(k) = \frac{B(k)}{A(k)} \qquad \text{and} \qquad m = MH,$$

where the sectionally meromorphic function H is defined by

$$H_1 = \begin{pmatrix} A & 0 \\ 0 & 1/A \end{pmatrix}, \qquad H_2 = H_3 = I, \qquad H_4 = \begin{pmatrix} 1/\bar{A} & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

Then, we formulate the new RH problem with the jump functions:

$$v_1 = \begin{pmatrix} \frac{a-\lambda b\bar{\Gamma}-\lambda\Gamma(\bar{a}\bar{\Gamma}-\bar{b})e^{2ikL}}{a-\lambda b\bar{\Gamma}} & -\Gamma e^{2ikL}e^{-2i\theta} \\ \frac{\lambda\bar{\Gamma}e^{2i\theta}}{(a-\lambda b\bar{\Gamma})(a+\lambda\bar{b}\Gamma e^{2ikL})} & \frac{a}{a+\lambda\bar{b}\Gamma e^{2ikL}} \end{pmatrix}, \qquad \arg k = \frac{\pi}{2},$$

$$v_2 = \begin{pmatrix} 1-\lambda\Gamma\bar{\Gamma} & -\frac{(\bar{a}\Gamma e^{2ikL}+b)e^{-2i\theta}}{\bar{a}+\lambda b\bar{\Gamma}e^{-2ikL}} \\ \frac{\lambda(a\bar{\Gamma}e^{-2ikL}+\bar{b})e^{2i\theta}}{a+\lambda\bar{b}\Gamma e^{2ikL}} & \frac{1}{(a+\lambda\bar{b}\Gamma e^{2ikL})(\bar{a}+\lambda b\bar{\Gamma}e^{-2ikL})} \end{pmatrix}, \qquad \arg k = 0,$$

$$v_3 = \begin{pmatrix} \frac{\bar{a}-\lambda\bar{b}\Gamma-\lambda\bar{\Gamma}(a\Gamma-b)e^{-2ikL}}{\bar{a}-\lambda\bar{b}\Gamma} & -\frac{\Gamma e^{-2i\theta}}{(\bar{a}-\lambda\bar{b}\Gamma)(\bar{a}+\lambda b\bar{\Gamma}e^{-2ikL})} \\ \lambda\bar{\Gamma}e^{-2ikL}e^{2i\theta} & \frac{\bar{a}}{\bar{a}+\lambda b\bar{\Gamma}e^{-2ikL}} \end{pmatrix}, \quad \arg k = -\frac{\pi}{2},$$

$$v_4 = \begin{pmatrix} \frac{1-\lambda\Gamma\bar{\Gamma}}{(a-\lambda b\bar{\Gamma})(\bar{a}-\lambda\bar{b}\Gamma)} & -\frac{(a\Gamma-b)e^{-2i\theta}}{\bar{a}-\lambda\bar{b}\Gamma} \\ \frac{\lambda(\bar{a}\bar{\Gamma}-\bar{b})e^{2i\theta}}{a-\lambda b\bar{\Gamma}} & 1 \end{pmatrix}, \qquad \arg k = \pi.$$

THE FINAL RH PROBLEM

We define the function $\tilde{\Gamma}(k)$ by

$$\tilde{\Gamma} = \frac{\lambda}{2e^{ikL}\bar{b}}\Big(\bar{a}e^{ikL} - ae^{-ikL} - i\sqrt{4-\Delta^2}\Big), \qquad (10)$$

where $\Delta(k)$ is given by

$$\Delta = ae^{-ikL} + \bar{a}e^{ikL}, \qquad k \in \mathbb{C}. \qquad (11)$$

We introduce the transform

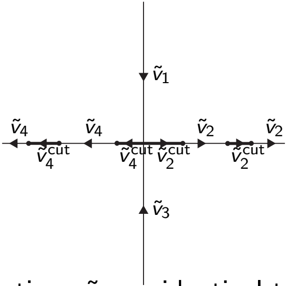
$$\tilde{m} = mg, \qquad (12)$$

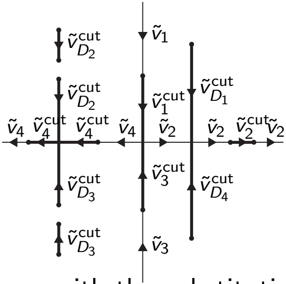
where the function $g(k)$ is given by

$$g_1 = \begin{pmatrix} \frac{a+\lambda\bar{b}\Gamma e^{2ikL}}{a+\lambda\bar{b}\tilde{\Gamma}e^{2ikL}} & (\tilde{\Gamma}-\Gamma)e^{2ikL}e^{-2i\theta} \\ 0 & \frac{a+\lambda\bar{b}\tilde{\Gamma}e^{2ikL}}{a+\lambda\bar{b}\Gamma e^{2ikL}} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & 0 \\ \frac{\lambda(\bar{\tilde{\Gamma}}-\bar{\Gamma})e^{2i\theta}}{(a-\lambda b\bar{\Gamma})(a-\lambda b\bar{\tilde{\Gamma}})} & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & \frac{(\tilde{\Gamma}-\Gamma)e^{-2i\theta}}{(\bar{a}-\lambda\bar{b}\Gamma)(\bar{a}-\lambda\bar{b}\tilde{\Gamma})} \\ 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} \frac{\bar{a}+\lambda b\bar{\tilde{\Gamma}}e^{-2ikL}}{\bar{a}+\lambda b\bar{\Gamma}e^{-2ikL}} & 0 \\ \lambda(\bar{\tilde{\Gamma}}-\bar{\Gamma})e^{-2ikL}e^{2i\theta} & \frac{\bar{a}+\lambda b\bar{\Gamma}e^{-2ikL}}{\bar{a}+\lambda b\bar{\tilde{\Gamma}}e^{-2ikL}} \end{pmatrix},$$

where g_j denotes the restriction of g to D_j for $j = 1, \dots, 4$.





The functions $\tilde{v}_j$ are identical to the v_j, with the substitution Γ to $\tilde{\Gamma}$, and

$$\begin{cases} \tilde{v}_{D_1}^{\text{cut}} = \begin{pmatrix} \frac{a+\lambda\bar{b}\tilde{\Gamma}_+ e^{2ikL}}{a+\lambda\bar{b}\tilde{\Gamma}_- e^{2ikL}} & (\tilde{\Gamma}_- - \tilde{\Gamma}_+)e^{2ikL}e^{-2i\theta} \\ 0 & \frac{a+\lambda\bar{b}\tilde{\Gamma}_- e^{2ikL}}{a+\lambda\bar{b}\tilde{\Gamma}_+ e^{2ikL}} \end{pmatrix}, \\ \tilde{v}_{D_2}^{\text{cut}} = \begin{pmatrix} 1 & 0 \\ \frac{\lambda(\bar{\tilde{\Gamma}}_- - \bar{\tilde{\Gamma}}_+)e^{2i\theta}}{(a-\lambda b\bar{\tilde{\Gamma}}_-)(a-\lambda b\bar{\tilde{\Gamma}}_+)} & 1 \end{pmatrix}, \\ \tilde{v}_{D_3}^{\text{cut}} = \begin{pmatrix} 1 & \frac{(\tilde{\Gamma}_- - \tilde{\Gamma}_+)e^{-2i\theta}}{(\bar{a}-\lambda\bar{b}\tilde{\Gamma}_-)(\bar{a}-\lambda\bar{b}\tilde{\Gamma}_+)} \\ 0 & 1 \end{pmatrix}, \\ \tilde{v}_{D_4}^{\text{cut}} = \begin{pmatrix} \frac{\bar{a}+\lambda b\bar{\tilde{\Gamma}}_- e^{-2ikL}}{\bar{a}+\lambda b\bar{\tilde{\Gamma}}_+ e^{-2ikL}} & 0 \\ \lambda(\bar{\tilde{\Gamma}}_- - \bar{\tilde{\Gamma}}_+)e^{2i\theta}e^{-2ikL} & \frac{\bar{a}+\lambda b\bar{\tilde{\Gamma}}_+ e^{-2ikL}}{\bar{a}+\lambda b\bar{\tilde{\Gamma}}_- e^{-2ikL}} \end{pmatrix}, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{v}_1^{\text{cut}} = \tilde{v}_{D_1}^{\text{cut}}\tilde{v}_{1-}, \\ \tilde{v}_2^{\text{cut}} = \tilde{v}_{D_1}^{\text{cut}}\tilde{v}_{2-}, \\ \tilde{v}_3^{\text{cut}} = \tilde{v}_{D_3}^{\text{cut}}\tilde{v}_{3-}, \\ \tilde{v}_4^{\text{cut}} = \tilde{v}_{D_3}^{\text{cut}}\tilde{v}_{4-}, \end{cases}$$

AN EXPLICIT EXAMPLE

Initial data:

$$q(x,0) = q_0 e^{\frac{2i\pi N}{L}x}, \qquad q_0 > 0, \quad N \in \mathbb{Z}, \qquad x \in [0,L]. \qquad (13)$$

Then, via the analysis of the RH we obtain the solution

$$q(x,t) = q_0 e^{\frac{2i\pi N}{L}x} e^{-2i\lambda q_0^2 t - \frac{4i\pi^2 N^2}{L^2}t}. \qquad (14)$$

The spectral functions a and b:

$$a(k) = \frac{e^{i(kL+\pi N)}(Lr\cos(Lr) - i(kL+\pi N)\sin(Lr))}{Lr}, \; b(k) = -\frac{q_0 e^{i(kL+\pi N)}\sin(Lr)}{r},$$

where $r(k)$ denotes the square root

$$r(k) = \sqrt{\left(k + \frac{\pi N}{L}\right)^2 - \lambda q_0^2}.$$

It follows that

$$\Delta(k) = 2(-1)^N \cos(Lr(k)).$$

The periodic spectrum is given by the two simple zeros

$$\lambda^{\pm} = \begin{cases} -\frac{\pi N}{L} \pm q_0 & \text{if } \lambda = 1, \\ -\frac{\pi N}{L} \pm i q_0 & \text{if } \lambda = -1. \end{cases}$$

Δ also has the following infinite sequence of double zeros

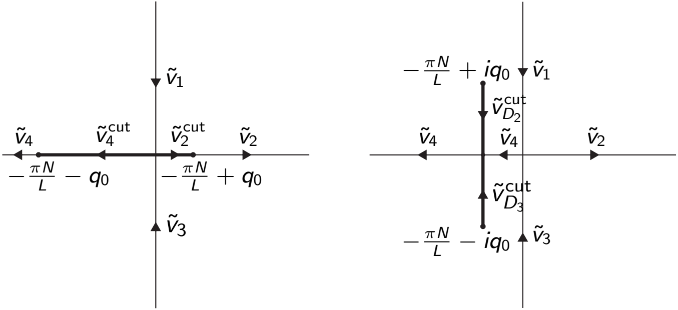
$$-\frac{\pi N}{L} \pm \sqrt{\frac{n^2\pi^2}{L^2} + \lambda q_0^2}, \qquad n \in \mathbb{Z} \setminus \{0\}.$$

Thus,

$$\sqrt{4 - \Delta^2} = 2(-1)^N \sin(Lr) \qquad \text{and} \qquad \tilde{\Gamma}(k) = -\frac{\lambda i(k - r + \frac{\pi N}{L})}{q_0}.$$

THE RH PROBLEM

For $N \geq 1$, the RH problem can be formulated as follows.



The jump contour and jump matrices for $\lambda = 1$ (left) and $\lambda = -1$ (right).

THE JUMP FUNCTIONS

$$\tilde{v}_1 = \begin{pmatrix} \frac{2\lambda r(k-r+\frac{\pi N}{L})}{q_0^2} & \frac{i\lambda(k-r+\frac{\pi N}{L})e^{-2i(\theta-kL)}}{q_0} \\ \frac{i(k-r+\frac{\pi N}{L})e^{2i(\theta-kL+Lr)}}{q_0} & \frac{(k+\frac{\pi N}{L})(1-e^{2iLr})+r(1+e^{2iLr})}{2r} \end{pmatrix}, \qquad k \in i\mathbb{R}_+,$$

$$\tilde{v}_2 = \begin{pmatrix} \frac{2\lambda r(k-r+\frac{\pi N}{L})}{q_0^2} & \frac{i\lambda(k-r+\frac{\pi N}{L})e^{-2i(\theta-kL+Lr)}}{q_0} \\ \frac{i(k-r+\frac{\pi N}{L})e^{2i(\theta-kL+Lr)}}{q_0} & 1 \end{pmatrix}, \qquad k \in \mathbb{R}_+ \setminus \mathcal{C},$$

$$\tilde{v}_3 = \begin{pmatrix} \frac{2\lambda r(k-r+\frac{\pi N}{L})}{q_0^2} & \frac{i\lambda(k-r+\frac{\pi N}{L})e^{-2i(\theta-kL+Lr)}}{q_0} \\ \frac{i(k-r+\frac{\pi N}{L})e^{2i(\theta-kL)}}{q_0} & \frac{(k+\frac{\pi N}{L})(1-e^{-2iLr})+r(1+e^{-2iLr})}{2r} \end{pmatrix}, \qquad k \in i\mathbb{R}_-,$$

$$\tilde{v}_4 = \begin{pmatrix} \frac{2\lambda r(k-r+\frac{\pi N}{L})}{q_0^2} & \frac{i\lambda(k-r+\frac{\pi N}{L})e^{-2i(\theta-kL)}}{q_0} \\ \frac{i(k-r+\frac{\pi N}{L})e^{2i(\theta-kL)}}{q_0} & 1 \end{pmatrix}, \qquad k \in \mathbb{R}_- \setminus \mathcal{C}.$$

Let $\mathfrak{r}(k) = \sqrt{|(k+\pi N/L)^2 - \lambda q_0^2|} \geq 0, \qquad \lambda \pm 1.$

For $\lambda = 1$ (left column), and $\lambda = -1$ (right column):

$$\tilde{v}_2^{\text{cut}} = \begin{pmatrix} 0 & \frac{i(k-i\mathfrak{r}(k)+\frac{\pi N}{L})e^{-2i(\theta-kL)}}{q_0} \\ \frac{i(k+i\mathfrak{r}(k)+\frac{\pi N}{L})e^{2i(\theta-kL)}}{q_0} & e^{-2L\mathfrak{r}(k)} \end{pmatrix}, \quad \tilde{v}_{D_2}^{\text{cut}} = \begin{pmatrix} 1 & 0 \\ \frac{2i\mathfrak{r}(k)e^{2i(\theta-kL)}}{q_0} & 1 \end{pmatrix},$$

$$\tilde{v}_4^{\text{cut}} = \begin{pmatrix} 0 & \frac{i(k+i\mathfrak{r}(k)+\frac{\pi N}{L})e^{-2i(\theta-kL)}}{q_0} \\ \frac{i(k-i\mathfrak{r}(k)+\frac{\pi N}{L})e^{2i(\theta-kL)}}{q_0} & 1 \end{pmatrix}, \quad \tilde{v}_{D_3}^{\text{cut}} = \begin{pmatrix} 1 & -\frac{2i\mathfrak{r}(k)e^{-2i(\theta-kL)}}{q_0} \\ 0 & 1 \end{pmatrix}.$$